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Kappeler, T ; Serier, F ; Topalov, P

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# On the symplectic phase space of KdV

T. Kappeler\*, F. Serier, and P. Topalov

## Abstract

We prove that the Birkhoff map  $\Omega$  for KdV constructed on  $H_0^{-1}(\mathbb{T})$  can be interpolated between  $H_0^{-1}(\mathbb{T})$  and  $L_0^2(\mathbb{T})$ . In particular, the symplectic phase space  $H_0^{1/2}(\mathbb{T})$  can be described in terms of Birkhoff coordinates.

## 1 Introduction

In [12] it is shown that the Birkhoff map for the Korteweg - de Vries equation (KdV), on the circle  $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ , introduced and studied in detail in [9, 6] can be analytically extended to an analytic diffeomorphism

$$\Omega : H_0^{-1}(\mathbb{T}) \rightarrow \mathfrak{h}^{-1/2}$$

from the Sobolev space of distributions  $H_0^{-1}(\mathbb{T})$  (dual of  $H_0^1(\mathbb{T})$ ) to the Hilbert space of sequences  $\mathfrak{h}^{-1/2}$  where for any  $\alpha \in \mathbb{R}$ ,

$$\mathfrak{h}^\alpha := \{z = (x_k, y_k)_{k \geq 1} \mid \|z\|_\alpha < \infty\},$$

with

$$\|z\|_\alpha := \left( \sum_{k \geq 1} k^{2\alpha} (x_k^2 + y_k^2) \right)^{1/2}.$$

In this paper we show that  $\Omega$  can be interpolated between  $H_0^{-1}(\mathbb{T})$  and  $L_0^2(\mathbb{T})$ .

**Theorem 1.** *For any  $-1 \leq \alpha \leq 0$ ,*

$$\Omega|_{H_0^\alpha(\mathbb{T})} : H_0^\alpha(\mathbb{T}) \rightarrow \mathfrak{h}^{\alpha+1/2}$$

*is a real analytic diffeomorphism.*

As an application of Theorem 1 we characterize the regularity of a potential  $q \in H^{-1}(\mathbb{T})$  in terms of the decay of the gap lengths  $(\gamma_k)_{k \geq 1}$  of the periodic spectrum of Hill's operator  $-\frac{d^2}{dx^2} + q$  on the interval  $[0, 2]$ . More precisely, recall

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that the periodic spectrum of  $-\frac{d^2}{dx^2} + q$  on the interval  $[0, 2]$  is discrete. When listed in increasing order (with multiplicities) the eigenvalues  $(\lambda_k)_{k \geq 0}$  satisfy

$$\lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \dots$$

The gap lengths  $\gamma_k = \gamma_k(q)$  are then defined by

$$\gamma_k := \lambda_{2k} - \lambda_{2k-1} \quad (k \geq 1).$$

**Theorem 2.** *For any  $q \in H^{-1}(\mathbb{T})$  and any  $-1 \leq \alpha \leq 0$ , the potential  $q$  is in  $H^\alpha(\mathbb{T})$  if and only if  $(\gamma_k(q))_{k \geq 1} \in \mathfrak{h}^\alpha$ .*

In a subsequent paper we will use Theorem 1 to study the solutions of the KdV equation (see [2, 3], [14], [21]) in the symplectic phase space  $H_0^{-1/2}(\mathbb{T})$  introduced by Kuksin [16].

*Method of proof:* Theorem 2 can be shown to be a consequence of Theorem 1 and formulas relating the  $n$ 'th action variable  $I_n$  with the  $n$ 'th gap length  $\gamma_n$  and their asymptotics as  $n \rightarrow \infty$ . In view of results established in [12] the proof of Theorem 1 consists in showing that for any  $-1 < \alpha < 0$  the restriction of  $\Omega$  to  $H_0^\alpha(\mathbb{T})$ ,  $\Omega|_{H_0^\alpha(\mathbb{T})} : H_0^\alpha(\mathbb{T}) \rightarrow \mathfrak{h}^{\alpha+1/2}$ , is *onto*. Our method of proof combines a study of the Birkhoff map at the origin together with a strikingly simple deformation argument to show that the map  $\Omega|_{H_0^\alpha(\mathbb{T})} : H_0^\alpha(\mathbb{T}) \rightarrow \mathfrak{h}^{\alpha+1/2}$  is onto. More precisely it uses that (1),  $d_0\Omega_\alpha : H_0^\alpha(\mathbb{T}) \rightarrow \mathfrak{h}^{\alpha+1/2}$  is a linear isomorphism, (2), that the map  $\Omega : H_0^{-1}(\mathbb{T}) \rightarrow \mathfrak{h}^{-1/2}$  is a canonical bi-analytic diffeomorphism, and (3), that the Hamiltonian vector field defining the deformation is actually in  $L^2$ . The same method could also be used for the proof of analogous results for more general weighted Sobolev spaces. In a subsequent work we plan to apply our technique to the defocusing Nonlinear Schrödinger equation.

*Related work:* Theorem 1 improves on earlier results in [12] where it was shown that  $\Omega|_{H_0^\alpha(\mathbb{T})} : H_0^\alpha(\mathbb{T}) \rightarrow \mathfrak{h}^{\alpha+1/2}$  is a bianalytic diffeomorphism onto its image for any  $-1 < \alpha < 0$ . For partial results in this direction see also [18]. The statement of Theorem 2 adds to numerous results characterizing the regularity of a potential by the decay of the corresponding gap lengths – see e. g. [4], [7], [15], [17], [19] and references therein. However only a few results concern potentials in spaces of distributions – see [8], [15] (cf. also [12] and the references therein). In a first attempt we have tried to apply the most beautiful and most simple approach among all the papers cited, due to Pöschel [19], to our case. However his methods seem to fail if  $\alpha \leq -3/4$ .

The idea of using flows to prove that a map is onto is not new in this subject. It has been used e.g. by Pöschel and Trubowitz in their book [20] or, to give a more recent example, in work of Chelkak and Korotyaev [1]. More precisely, in [20, Theorem 2, p. 115], the authors use flows to characterize sequences coming up as sequences of Dirichlet eigenvalues of Schrödinger operators  $-\frac{d^2}{dx^2} + q$  on  $[0, 1]$  with an even  $L^2$ -potential  $q$ . Note however, that in this paper the

use of flows is of a different nature, best explained by the fact that they are regularizing - in other words, the vector fields describing the deformations are in a higher Sobolev space than the underlying phase space.

## 2 Proof of Theorem 1

Let  $\Omega$  be the Birkhoff map  $\Omega : H_0^{-1}(\mathbb{T}) \rightarrow \mathfrak{h}^{-1/2}$  constructed in [12] – see also Appendix for a brief summary of the results in [12]. By Theorem 3 in Appendix, the Birkhoff map  $\Omega$  is onto and for any given  $\alpha > -1$  its restriction to  $H_0^\alpha(\mathbb{T})$  is a map

$$\Omega_\alpha := \Omega|_{H_0^\alpha(\mathbb{T})} : H_0^\alpha(\mathbb{T}) \rightarrow \mathfrak{h}^{\alpha+1/2} \quad (1)$$

which is a bianalytic diffeomorphism onto its image. Hence, in order to prove Theorem 1 we need to prove that (1) is onto.

Assume that there exists  $-1 \leq \alpha \leq 0$  such that  $\Omega_\alpha : H_0^\alpha(\mathbb{T}) \rightarrow \mathfrak{h}^{\alpha+1/2}$  is *not* onto. As  $\Omega : H_0^{-1}(\mathbb{T}) \rightarrow \mathfrak{h}^{-1/2}$  is onto it then follows that there exists

$$q_0 \in H_0^{-1}(\mathbb{T}) \setminus H_0^\alpha(\mathbb{T}) \quad (2)$$

such that  $\Omega(q_0) \in \mathfrak{h}^{\alpha+1/2}$ .

As  $\Omega(0) = 0$  and as by Corollary 1 in the Appendix below the differential

$$d_0\Omega_\alpha : H_0^\alpha(\mathbb{T}) \rightarrow \mathfrak{h}^{\alpha+1/2}$$

of (1) at  $q = 0$  is a linear isomorphism, one gets from the inverse function theorem that there exist an open neighborhood  $U_\alpha$  of zero in  $H_0^\alpha(\mathbb{T})$  and an open neighborhood  $V_\alpha$  of zero in  $\mathfrak{h}^{\alpha+1/2}$  such that

$$\Omega|_{U_\alpha} : U_\alpha \rightarrow V_\alpha \quad (3)$$

is a diffeomorphism.

Recall that for any  $k \geq 1$  the angle variable  $\theta_k$  constructed in [12] is a real-analytic function on  $H_0^{-1}(\mathbb{T}) \setminus D_k$  with values in  $\mathbb{R}/2\pi\mathbb{Z}$  where  $D_k := \{q \in H_0^{-1}(\mathbb{T}) \mid \gamma_k(q) = 0\}$  is a real-analytic sub-variety in  $H_0^{-1}(\mathbb{T})$  (cf. Appendix). As  $\theta_k$  is real-analytic, the mapping  $H_0^{-1}(\mathbb{T}) \setminus D_k \rightarrow H_0^1(\mathbb{T})$ ,  $q \mapsto \frac{\partial \theta_k}{\partial q}(q)$ , is real-analytic<sup>1</sup> and therefore,

$$H_0^{-1}(\mathbb{T}) \setminus D_k \rightarrow L_0^2(\mathbb{T}), \quad q \mapsto Y_k(q) := \frac{d}{dx} \frac{\partial \theta_k}{\partial q}(q), \quad (4)$$

is real-analytic as well. Then  $Y_k$  is a Hamiltonian vector field on  $H_0^{-1}(\mathbb{T}) \setminus D_k$ , which defines a dynamical system

$$\dot{q} = Y_k(q), \quad q(0) = q_0 \in H_0^{-1}(\mathbb{T}) \setminus D_k. \quad (5)$$

Let  $q_0 \in H_0^{-1}(\mathbb{T}) \setminus D_k$  and assume that

$$\Omega(q_0) = (z_1^0, z_2^0, \dots) \in \mathfrak{h}^{\alpha+1/2}$$

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<sup>1</sup>  $\frac{\partial \theta_k}{\partial q}$  denotes the  $L^2$ -gradient of  $\theta_k$ .

where for any  $n \geq 1$ ,  $z_n^0 = (x_n^0, y_n^0)$ . Take  $\varepsilon > 0$  such that the ball

$$B(2\varepsilon) := \{z \in \mathfrak{h}^{\alpha+1/2} \mid \|z\|_{\alpha+1/2} < 2\varepsilon\}$$

is contained in the neighborhood  $V_\alpha$  of zero in  $\mathfrak{h}^{\alpha+1/2}$  chosen in (3). Denote by  $I_n = I_n(q)$  the  $n$ 'th action variable of a potential  $q$  – see Appendix. Note that for any  $q$  in  $H_0^{-1}(\mathbb{T})$

$$2I_n(q) = \|z_n(q)\|^2 = x_n(q)^2 + y_n(q)^2 \quad (6)$$

where  $\Omega(q) = (z_n(q))_{n \geq 1}$  and  $z_n(q) = (x_n(q), y_n(q))$ . Consider the sequence of potentials  $(q_n)_{n \geq 1}$  in  $H_0^{-1}(\mathbb{T})$  defined recursively for  $n \geq 1$  by

$$q_n := \begin{cases} q_{n-1} & \text{if } 2I_n(q_{n-1}) < \varepsilon/(n^{1+2\alpha} 2^n) \\ (q_{n-1})_{,n} & \text{otherwise} \end{cases}$$

where  $(q_{n-1})_{,n}$  is obtained by shifting  $q_{n-1}$  along the flow of the vector field  $Y_n$  such that

$$2I_n((q_{n-1})_{,n}) < \varepsilon/(n^{1+2\alpha} 2^n).$$

The existence of  $(q_{n-1})_{,n}$  follows from Lemma 1 (a) below. Moreover, by the commutator relations (19) in Appendix,

$$Y_n(I_m) = \{I_m, \theta_n\} = 0 \quad (n \neq m),$$

the vector field  $Y_n$  preserves the values of the action variables  $I_m$  for any  $m \neq n$ . In particular, we get

$$2I_j(q_n) \leq \varepsilon/(j^{1+2\alpha} 2^j), \quad \forall 1 \leq j \leq n \quad (7)$$

and

$$2I_j(q_n) = \|z_j^0\|^2, \quad \forall j > n. \quad (8)$$

One obtains from (7), (8), and  $\|z_j\|^2 = 2I_j$  (cf. (17)) that

$$\|\Omega(q_n)\|_{\alpha+1/2}^2 = \sum_{j=1}^{\infty} j^{1+2\alpha} \|z_j(q_n)\|^2 \leq \varepsilon \sum_{1 \leq j \leq n} \frac{1}{2^j} + \sum_{j \geq n+1} j^{1+2\alpha} \|z_j^0\|^2. \quad (9)$$

As  $\sum_{j \geq 1} j^{1+2\alpha} \|z_j^0\|^2 = \|\Omega(q_0)\|_{\alpha+1/2}^2 < \infty$ , one gets from (9) that there exists  $N \geq 1$  such that for any  $n \geq N$ ,  $\|\Omega(q_n)\|_{\alpha+1/2} < 2\varepsilon$ . In particular,  $\Omega(q_N) \in V_\alpha$  and, as  $\Omega|_{U_\alpha} : U_\alpha \rightarrow V_\alpha$  is a diffeomorphism, the bijectivity of the Birkhoff map  $\Omega : H_0^{-1} \rightarrow \mathfrak{h}^{-1/2}$  implies that

$$q_N \in U_\alpha \subseteq H_0^\alpha(\mathbb{T}). \quad (10)$$

On the other side, it follows from (2) and Lemma 1 (b) that

$$(q_n)_{n \geq 1} \subseteq H_0^{-1}(\mathbb{T}) \setminus H_0^\alpha(\mathbb{T})$$

which implies  $q_N \in H_0^{-1}(\mathbb{T}) \setminus H_0^\alpha(\mathbb{T})$ , contradicting (10). This completes the proof of Theorem 1.  $\square$

The following Lemma was used in the proof of Theorem 1.

**Lemma 1.** *For any  $k \geq 1$  and for any initial data  $q_0 \in H_0^{-1}(\mathbb{T}) \setminus D_k$  the initial value problem (5) has a unique solution in  $C^1((-I_k^0, \infty), H_0^{-1}(\mathbb{T}))$  where  $I_k^0 \geq 0$  is the value of the action variable  $I_k$  at  $q_0$ . The solution has the following additional properties:*

$$(a) \quad \lim_{t \rightarrow -I_k^0 + 0} I_k(q(t)) = 0;$$

$$(b) \quad q(t) - q_0 \in L_0^2(\mathbb{T}).$$

*Proof of Lemma 1.* By Theorem 3 in the Appendix, the Birkhoff map  $\Omega : H_0^{-1}(\mathbb{T}) \rightarrow \mathfrak{h}^{-1/2}$ ,

$$q \mapsto \Omega(q) = (z_1, z_2, \dots), \quad z_n = (x_n, y_n),$$

is a bianalytic diffeomorphism that transforms the Poisson structure  $\frac{d}{dx}$  on  $H_0^{-1}(\mathbb{T})$  (cf. Appendix) into the canonical Poisson structure on  $\mathfrak{h}^{-1/2}$  defined by the relations  $\{x_m, x_n\} = \{y_m, y_n\} = 0$  and  $\{x_m, y_n\} = \delta_{mn}$  that hold for any  $m, n \geq 1$ .<sup>2</sup> Moreover, it follows from the construction of the Birkhoff map  $\Omega$  that  $\theta_k$  is the argument of the complex number  $x_k + iy_k$ . In particular, in Birkhoff coordinates  $(z_1, z_2, \dots) \in \mathfrak{h}^{-1/2}$ , one has for any  $q \in H_0^{-1}(\mathbb{T}) \setminus D_k$

$$d\Omega(Y_k) = \frac{x_k}{x_k^2 + y_k^2} \frac{\partial}{\partial x_k} + \frac{y_k}{x_k^2 + y_k^2} \frac{\partial}{\partial y_k}. \quad (11)$$

The dynamical system corresponding to the vector field (11) in  $\mathfrak{h}^{-1/2}$  has a unique solution for any initial data  $(x_n^0, y_n^0)_{n \geq 1}$  that is defined on the time interval  $(-((x_k^0)^2 + (y_k^0)^2)/2, \infty)$ . Hence, as  $\Omega : H_0^{-1}(\mathbb{T}) \rightarrow \mathfrak{h}^{-1/2}$  is a diffeomorphism, the dynamical system (5) has a unique solution  $q(t)$  on  $H_0^{-1}(\mathbb{T}) \setminus D_k$  defined for  $t \in (-I_k(q_0), \infty)$ . Moreover, one gets from (11) and (6) that

$$\lim_{t \rightarrow -I_k(q_0) + 0} I_k(q(t)) = 0.$$

This completes the proof of (a). In order to prove (b) we integrate both sides of (5) in  $H_0^{-1}(\mathbb{T})$  and get that for any  $t \in (-\infty, I_k(q_0))$ ,

$$q(t) = q_0 + \int_0^t Y_k(q(s)) ds. \quad (12)$$

As the mapping (4) is real-analytic (and hence, continuous) and as the solution  $q(t)$  of (5) is a  $C^1$ -curve  $(-\infty, I_k(q_0)) \rightarrow H_0^{-1}(\mathbb{T})$ , the integrand in (12) is in  $C^0((-I_k(q_0), \infty), L_0^2(\mathbb{T}))$ . In particular, the integral in (12) converges with respect to the  $L^2$ -norm, and hence represents an element in  $L_0^2(\mathbb{T})$ . This proves (b).  $\square$

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<sup>2</sup>Here  $\delta_{mn}$  denotes the Kronecker delta.

### 3 Proof of Theorem 2

As for any constant  $c \in \mathbb{R}$ , the potentials  $q$  and  $q + c$  have the same sequence of gap lengths  $(\gamma_k)_{k \geq 1}$  it is enough to prove the statement of the theorem for  $q \in H_0^{-1}(\mathbb{T})$ .

For  $q \in H_0^{-1}(\mathbb{T})$  given let

$$z = (z_1, z_2, \dots) = \Omega(q),$$

where for any  $n \geq 1$ ,  $z_n = (x_n, y_n)$ . By Proposition 1 in Appendix, there exist constants  $0 < C_1 < C_2 < \infty$  and  $n_0 \geq 1$  depending on  $q$  such that for any  $n \geq n_0$ ,

$$C_1 \frac{\gamma_n^2}{n} \leq I_n \leq C_2 \frac{\gamma_n^2}{n} \quad (13)$$

where  $I_n$  is the  $n$ -th action variable of the given potential  $q$ . Using that

$$I_n = (x_n^2 + y_n^2)/2$$

we get from (13) that for any given  $\alpha \geq -1$ ,

$$(z_n)_{n \geq 1} \in \mathfrak{h}^{\alpha+1/2} \iff (\gamma_n)_{n \geq 1} \in \mathfrak{h}^\alpha. \quad (14)$$

On the other side, it follows from Theorem 1 and the injectivity of  $\Omega : H_0^{-1}(\mathbb{T}) \rightarrow \mathfrak{h}^{-1/2}$  that

$$(z_n)_{n \geq 1} \in \mathfrak{h}^{\alpha+1/2} \iff q \in H_0^\alpha(\mathbb{T}). \quad (15)$$

Theorem 2 now follows from (14) and (15).  $\square$

### 4 Appendix

In this appendix we collect the properties of the Birkhoff map  $\Omega : H_0^{-1}(\mathbb{T}) \rightarrow \mathfrak{h}^{-1/2}$  constructed in [12] that were used in the proofs of Theorem 1 and Theorem 2.

The Korteweg - de Vries equation (KdV)

$$\begin{aligned} q_t - 6qq_x + q_{xxx} &= 0 \\ q|_{t=0} &= q_0 \end{aligned}$$

on the circle can be viewed as an integrable PDE, i.e. an integrable Hamiltonian system of infinite dimension. As a phase space we consider the Sobolev space  $H^\alpha(\mathbb{T})$  ( $\alpha \geq -1$ ) of real valued distributions on the circle. The Poisson bracket is the one proposed by Gardner,

$$\{F, G\} := \int_{\mathbb{T}} \frac{\partial F}{\partial q} \frac{d}{dx} \left( \frac{\partial G}{\partial q} \right) dx \quad (16)$$

where  $F, G$  are  $C^1$ -functions on  $H^\alpha(\mathbb{T})$  and  $\frac{\partial F}{\partial q}, \frac{\partial G}{\partial q}$  denote the  $L^2$ -gradients of  $F$  and  $G$  respectively which are assumed to be sufficiently smooth so that the

Poisson bracket is well defined. For  $q$  sufficiently smooth, i.e.  $q \in H_0^1(\mathbb{T})$ , the Hamiltonian  $\mathcal{H}$  corresponding to KdV is given by

$$\mathcal{H}(q) = \int_{\mathbb{T}} ((\partial_x q)^2/2 + q^3) dx$$

and the KdV equation can be written in Hamiltonian form

$$q_t = \frac{d}{dx} \frac{\partial \mathcal{H}}{\partial q}.$$

Note that the Poisson structure is degenerate and admits the average  $[q] := \int_{\mathbb{T}} q(x) dx$  as a Casimir function. Moreover, the Poisson structure is regular and induces a trivial foliation whose leaves are given by

$$H_c^\alpha(\mathbb{T}) = \{q \in H^\alpha(\mathbb{T}) \mid [q] = c\}.$$

Introduce the set

$$D_k := \{q \in H_0^{-1}(\mathbb{T}) \mid \gamma_k(q) = 0\}.$$

For any  $q \in H_0^{-1}(\mathbb{T}) \setminus D_k$  define

$$z_k(q) := \sqrt{2I_k(q)} \left( \cos(\theta_k(q)), \sin(\theta_k(q)) \right), \quad (17)$$

where  $I_k(q)$  is the  $k$ 'th action variable and  $\theta_k(q)$  is the  $k$ 'th angle variable of the KdV equation (cf. § 3, 4 in [12]). It is shown in [12, § 5] that the mapping  $H_0^{-1} \setminus D_k \rightarrow \mathbb{R}^2$ ,  $q \mapsto z_k(q)$ , extends analytically to  $H_0^{-1}(\mathbb{T})$ . For any  $q \in H_0^{-1}(\mathbb{T})$  the action variables  $(I_k)_{k \geq 1}$  of KdV are defined in terms of the periodic spectrum of the Schrödinger operator  $-\frac{d^2}{dx^2} + q$  using the same formulas as in [5] (cf. also [9]). For any given  $\alpha \geq -1$  and for any  $k \geq 1$  the action  $I_k$  is a real analytic function on  $H_0^\alpha(\mathbb{T})$  (cf. Proposition 3.3 in [12]). The angle  $\theta_k$  is defined modulo  $2\pi$  and is a real analytic function on  $H_0^\alpha(\mathbb{T}) \setminus (D_k \cap H_0^\alpha)$ , where  $D_k \cap H_0^\alpha = \{q \in H_0^\alpha(\mathbb{T}) \mid \gamma_k(q) = 0\}$  is a real analytic sub-variety in  $H_0^\alpha(\mathbb{T})$  of co-dimension two (cf. Proposition 4.3 in [12]). By § 6 in [12] we have the following commutator relations

$$\{I_m, I_n\} = 0 \text{ on } H_0^{-1}(\mathbb{T}) \quad (18)$$

$$\{I_m, \theta_n\} = \delta_{nm} \text{ on } H_0^{-1}(\mathbb{T}) \setminus D_n \quad (19)$$

and

$$\{\theta_m, \theta_n\} = 0 \text{ on } H_0^{-1}(\mathbb{T}) \setminus (D_m \cup D_n) \quad (20)$$

for any  $m, n \geq 1$ . For any  $q \in H_0^{-1}(\mathbb{T})$  define

$$\Omega(q) := (z_1(q), z_2(q), \dots)$$

where  $z_k = z_k(q)$  is given by (17). It is shown in [12] that  $\Omega(q) \in \mathfrak{h}^{-1/2}$ . Recall that, for any  $\alpha \in \mathbb{R}$ ,  $\mathfrak{h}^\alpha$  denotes the Hilbert space

$$\mathfrak{h}^\alpha = \{z = (x_k, y_k)_{k \geq 1} \mid \|z\|_\alpha < \infty\},$$



with the norm

$$\|z\|_\alpha := \left( \sum_{k \geq 1} k^{2\alpha} (x_k^2 + y_k^2) \right)^{1/2}.$$

We supply  $\mathfrak{h}^{-1/2}$  with a Poisson structure defined by the relations  $\{x_m, x_n\} = \{y_m, y_n\} = 0$  and  $\{x_m, y_n\} = \delta_{mn}$  valid for any  $m, n \geq 1$ . The following result is proved in [12].

**Theorem 3.** *The mapping  $\Omega : H_0^{-1}(\mathbb{T}) \rightarrow \mathfrak{h}^{-1/2}$  satisfies the following properties:*

- (i)  $\Omega$  is a bianalytic diffeomorphism that preserves the Poisson bracket;
- (ii) for any  $\alpha > -1$ , the restriction  $\Omega_\alpha \equiv \Omega|_{H_0^\alpha(\mathbb{T})}$  is a map  $\Omega|_{H_0^\alpha(\mathbb{T})} : H_0^\alpha(\mathbb{T}) \rightarrow \mathfrak{h}^{\alpha+1/2}$  which is one-to-one and bianalytic onto its image. In particular, the image is an open subset in  $\mathfrak{h}^{\alpha+1/2}$ .

**Corollary 1.** *For any  $\alpha > -1$ ,*

$$d_0 \Omega_\alpha : H_0^\alpha(\mathbb{T}) \rightarrow \mathfrak{h}^{\alpha+1/2},$$

*is a linear isomorphism.*

We will also need the following Proposition (cf. [12, § 3]).

**Proposition 1.** *There exists a complex neighborhood  $\mathcal{W}$  of  $H_0^{-1}(\mathbb{T})$  in the complex space  $H_0^{-1}(\mathbb{T}, \mathbb{C})$  such that the quotient  $I_n/\gamma_n^2$ , defined on  $H_0^{-1}(\mathbb{T}) \setminus D_n$ , extends analytically to  $\mathcal{W}$  for all  $n$ . Moreover, for any  $\varepsilon > 0$  and any  $p \in \mathcal{W}$  there exists  $n_0 \geq 1$  and an open neighborhood  $U(p)$  of  $p$  in  $\mathcal{W}$  so that*

$$\left| 8\pi n \frac{I_n}{\gamma_n^2} - 1 \right| \leq \varepsilon$$

*for any  $n \geq n_0$  and for any  $q \in U(p)$ .*

Further we recall that for any  $q \in H_0^{-1}(\mathbb{T})$  one has that  $I_n(q) = 0$  if and only if  $\gamma_n(q) = 0$ . In particular, one concludes from (17) and the fact  $\gamma_n(0) = 0 \forall n \geq 1$  that  $\Omega(0) = 0$ .

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